Fourier Transform Applications

The Fourier transform is very useful in solving a variety of linear constant coefficient ordinary and partial differential equations describing processes which take place over an infinite interval, $-\infty < x < \infty$. We will provide a number of examples of this sort of application in the present section

Example 1: Heat Conduction in an Infinite Rod We consider an infinite bar of heat conducting material, parametrized by the coordinate x, $-\infty < x < \infty$. We will suppose the specific heat per unit length is the constant σ and the heat conductivity coefficient is κ. Denoting the temperature at the point x and time t by $u(x,t)$, it may be shown that the partial differential equation regulating the evolution of this temperature distribution is

$$\sigma \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

We will suppose that at time $t = 0$ the initial temperature distribution is $u(x,0) = u_0(x) \in L^2(-\infty,\infty)$. Applying the Fourier transform to both sides of the partial differential equation, and using the **differentiation property** of the Fourier transform, we have, for $\hat{u}(\xi,t) = (\mathcal{F}u(x,t))(\xi)$,

$$\frac{\partial \hat{u}}{\partial t} = \frac{\kappa}{\sigma} \left(\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}(x,t) \right) \right)(\xi) = -\frac{\kappa\,\xi^2}{\sigma} \hat{u}(\xi,t).$$

This is a first order linear differential equation, parametrized by ξ, in $\hat{u}(\xi,t)$. Taking account of the given initial distribution we have

$$\hat{u}(\xi,t) = e^{-\frac{\kappa\xi^2 t}{\sigma}}\,\hat{u_0}(\xi),\ \ t \geq 0,\ \ -\infty < \xi < \infty,$$

where $\hat{u_0}(\xi)$ is the Fourier transform of the initial state $u_0(x)$. Next applying the inverse Fourier transform to recover $u(x,t)$ we obtain, using the **convolution property** (which operates in the same way for the inverse Fourier

transform as it does for the transform itself)

$$u(x,t) \;=\; \sqrt{2\pi}\left(\mathcal{F}^{-1}\left(e^{-\frac{\kappa\xi^2 t}{\sigma}}\right)\right)(x) * u_0(x)$$

$$= \left(\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-\frac{\kappa\xi^2 t}{\sigma}}\, d\xi\right) \,*\, u_0(x).$$

For the evaluation of the integral we proceed as in Example 1 in the **Fourier transform properties** section. We have, for $t \,>\, 0$,

$$\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-\frac{\kappa\xi^2 t}{\sigma}}\, d\xi \;=\; e^{-\frac{\sigma x^2}{4\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{\kappa t}{\sigma}(\xi-(\sigma i x/2\kappa t))^2}\, d\xi$$

$$= e^{-\frac{\sigma x^2}{4\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{\kappa t}{\sigma}\xi^2}\, d\xi \;=\; e^{-\frac{\sigma x^2}{4\kappa t}} \left(\int_0^{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\kappa t}{\sigma}r^2}\, r\, dr\, d\theta\right)^{\frac{1}{2}}$$

$$= e^{-\frac{\sigma x^2}{4\kappa t}} \left(\frac{\pi\sigma}{\kappa t}\int_0^{r} e^{-\frac{\kappa t}{\sigma}r^2}\,\frac{2\kappa t}{\sigma}\, r\, dr\, d\theta\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{\pi\sigma}{\kappa t}}\, e^{-\frac{\sigma x^2}{4\kappa t}}.$$

Then we have the solution $u(x,t),\;\; t \,>\, 0$, in the form of the convolution integral

$$u(x,t) \;=\; \left(\int_{-\infty}^{\infty} e^{i\xi x}\, e^{-\frac{\kappa\xi^2 t}{\sigma}}\, d\xi\right) \,*\, u_0(x) \;=\; \sqrt{\frac{\pi\sigma}{\kappa t}} \int_{-\infty}^{\infty} e^{-\frac{\sigma y^2}{4\kappa t}}\, u_0(x-y)\, dy.$$

Example 2: Solution of the Wave Equation Lateral vibrations of a stretched string, or cable, of effectively infinite length (what that means would clearly depend on the context) can be modelled by the linear partial differential equation, known as the *wave equation*,

$$\rho\,\frac{\partial^2 u}{\partial t^2} \,-\, \tau\,\frac{\partial^2 u}{\partial x^2} \;=\; f(x,t),$$

where ρ is the mass per unit length, tau is the tension and $f(x,t)$ is an applied external lateral force distribution, all measured in appropriate units

which we do not need to specify here. We will treated the homogeneous, unforced, case wherein $f(x,t) \equiv 0$. The solution $u(x,t)$ is then completely determined by the initial conditions

$$u(x,0) = u_0(x), \ \frac{\partial u}{\partial t}(x,0) = v_0(x), \ -\infty < x < \infty,$$

where $u_0(x) \in L^2(-\infty,\infty)$ is the initial lateral displacement of the string and $v_0(x) \in L^2(-\infty,\infty)$ is the initial velocity in the lateral direction (in reality there would ordinarily be two lateral directions; we are modelling motion in just one of these). For the *energy integral*

$$\mathcal{E} = \frac{1}{2}\int_{-\infty}^{\infty}\left(\rho\,\frac{\partial u}{\partial t}^2 + \tau\,\frac{\partial u}{\partial x}^2\right)dx$$

to be finite we would also need to require that $u_0'(x) \in L^2(-\infty,\infty)$ but that will not play any direct role here.

Letting $\hat{u}(\xi,t)$, $\hat{u_0}(\xi)$ and $\hat{v_0}(\xi)$ denote the Fourier transforms of $u(x,t)$, $u_0(x)$ and $v_0(x)$, respectively, application of the transform to the partial differential equation, together with use of the **differentiation property** of the transform, yields

$$\rho\,\frac{\partial^2\hat{u}}{\partial t^2} + \tau\,\xi^2\,\hat{u} = 0.$$

This is a first order linear ordinary differential equation in $\hat{u}(\xi,t)$, parametrized by ξ. The general solution is

$$\hat{u}(\xi,t) = c_1(\xi)\,\cos\sigma\xi t + c_2(\xi)\,\sin\sigma\xi t,$$

where $\sigma = \sqrt{\frac{\tau}{\rho}}$. Matching this solution form to the transforms $\hat{u_0}(\xi)$, $\hat{v_0}(\xi)$ of the initial data we obtain $c_1(\xi) = \hat{u_0}(\xi)$, $c_2(\xi) = \frac{\hat{v_0}(\xi)}{\sigma\xi}$, so that

$$\hat{u}(\xi,t) = \hat{u_0}(\xi)\,\cos\sigma\xi t + \frac{\hat{v_0}(\xi)}{\sigma\xi}\,\sin\sigma\xi t.$$

Now

$$\cos\sigma\xi t\,\hat{u}_0(\xi) \;=\; \frac{1}{2}\left(e^{-i\sigma\xi t} \;+\; e^{-i\sigma\xi(-t)}\right)\,\hat{u}_0(\xi)$$

$$=\;\frac{1}{2}\left(\left(\mathcal{F}u_0(x-\sigma t)\right)(\xi)\;+\;\left(\mathcal{F}u_0(\xi+\sigma t)\right)(\xi)\right).$$

On the other hand $\frac{\sin\sigma\xi t}{\sigma\xi}$ is the Fourier transform of the function $\sqrt{2\pi}\,\frac{1}{2\sigma}\,\chi_{[-\sigma t,\sigma t]}(x)$, where

$$\chi_{[-\sigma t,\sigma t]}(x) \;=\; \begin{cases} 1, & |x| \,\le\, \sigma t, \\ 0, & \text{otherwise,} \end{cases}$$

as we can see from

$$\int_{-\sigma t}^{\sigma t}\frac{1}{2\sigma}\,e^{-i\xi x}\,dx \;=\; -\,\frac{e^{-i\xi x}}{2i\sigma\xi}\Big|_{-\sigma t}^{\sigma t}$$

$$=\;\frac{1}{2i\sigma\xi}\left(e^{i\xi\sigma t}\;-\;e^{-i\xi\sigma t}\right)\;=\;\frac{\sin\sigma\xi t}{\sigma\xi}.$$

We conclude, using the **convolution property** of the Fourier transform, that $\frac{\hat{v}_0(\xi)}{\sigma\xi}\,\sin\sigma\xi t$ is the Fourier transform of the convolution

$$\frac{1}{2\sigma}\,\chi_{[-\sigma t,\sigma t]}(x) * v_0(x) \;=\; \frac{1}{2\sigma}\int_{-\sigma t}^{\sigma t} v_0(x-y)\,dy.$$

Combining these results we see that

$$\hat{u}(\xi,t) \;=\; \frac{1}{2}\Big(\left(\mathcal{F}u_0(x-\sigma t)\right)(\xi)\;+\;\left(\mathcal{F}u_0(\xi+\sigma t)\right)(\xi)\Big)$$

$$+\;\left(\mathcal{F}\left(\frac{1}{2\sigma}\int_{-\sigma t}^{\sigma t} v_0(x-y)\,dy\right)\right)(\xi)$$

and, applying the inverse transform, we have the formula for the solution:

$$u(x,t) \;=\; \frac{1}{2}\Big(u_0(x-\sigma t)\;+\;u_0(\xi+\sigma t)\Big)\;+\;\frac{1}{2\sigma}\int_{-\sigma t}^{\sigma t} v_0(x-y)\,dy.$$

We should remark that it is also possible to obtain this formula without resort to the Fourier transform by using the so-called *method of characteristics*. This

result is, as a consequence, somewhat less convincing than that of Example 1, which would be rather difficult to obtain by any other method.

We have thus far avoided making any direct use of contour integration, all the while warning the reader that this technique is really quite indispensable for effective evaluation of the inverse transform. In our final example for this section we will show what can be done if contour integration is admitted.

Example 3: The Curious Case of the Elastically Suspended Catwalk

Let us suppose that a catwalk, which we model as an elastic beam, is suspended from a rigid support, say a very strong steel girder, by means of elastic stringers, or springs. Assuming the distance traversed is quite long with respect to any other dimensions involved, we can do the modelling on an infinite interval $-\infty < x < \infty$. We will suppose the catwalk supports a point mass m at $x = 0$ and the gravitational constant is g. Introducing the distribution δ_0, as in the section on **Laplace transforms of distributions**, the vertical deflection $u(x)$ of the catwalk from its nominal equilibrium level can be modelled by

$$EI\,\frac{\partial^4 u}{\partial x^4} \;=\; -\,\kappa\, u \;-\; mg\,\delta_0,$$

where EI is the *bending moment* of the elastic beam and κ is the stiffness, per unit beam length, of the elastic supporting structure.

Applying the Fourier transform, using its **differentiation property** and recognizing that, with the Fourier transform as we have defined it, the transform of δ_0 is the constant function $\frac{1}{\sqrt{2\pi}}$, we have

$$EI\,\xi^4 \hat{u}(\xi) \;+\; k\,\hat{u}(\xi) \;=\; -\,mg \;\;\Rightarrow\;\; \hat{u}(\xi) \;=\; -\,\frac{mg}{EI\sqrt{2\pi}}\,\frac{1}{\xi^4 + a^4},$$

where $a = \sqrt[4]{\frac{k}{EI}}$. Then, in order to take the inverse transform and identify

$u(x)$, we need to compute the integral

$$u(x) \;=\; -\frac{mg}{2EI\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi^4+a^4}\,d\xi.$$

We factor the denominator of the main fraction in the integrand

$$\xi^4 \,+\, a^4 \;=\; (\xi^2+i\,a^2)(\xi^2-i\,a^2)$$

$$= \left(\xi-\frac{1+i}{\sqrt{2}}a\right)\left(\xi-\frac{1-i}{\sqrt{2}}a\right)\left(\xi+\frac{1+i}{\sqrt{2}}a\right)\left(\xi+\frac{1-i}{\sqrt{2}}a\right).$$

Combining the polynomial decay of the expression $\frac{1}{\xi^4+a^4}$ as $|\xi| \,\to\, \infty$ with the exponential decay of $\left|e^{i\xi x}\right| \;=\; \left|e^{i(\eta+i\omega)x}\right| \;=\; e^{-\omega x}$ as $\omega \,\to\, +\infty,\;\; x \,>\, 0$, and as $\omega \,\to\, -\infty$, $x \,<\, 0$, we can apply the **calculus of residues** in complex variables theory to obtain the following

Result 1 *For $x \,>\, 0$ the integral $\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi^4+a^4}\,d\xi$ has the value $2\pi i\sum\frac{e^{i\zeta x}}{\frac{d}{d\xi}(\xi^4+a^4)\Big|_{\xi=\zeta}}$, where the sum is taken over values of ζ in the upper half complex plane for which $\zeta^4+a^4 \,=\, 0$. For $x \,<\, 0$ the integral has the value $-2\pi i\sum\frac{e^{i\zeta x}}{\frac{d}{d\xi}(\xi^4+a^4)\Big|_{\xi=\zeta}}$, where the sum is taken over values of ζ in the lower half complex plane for which $\zeta^4+a^4 \,=\, 0$.*

From the factorization we conclude that the ξ^4+a^4 has simple zeros at the points $\zeta \,=\, \frac{1+i}{\sqrt{2}}a$ and $\zeta \,=\, \frac{-1+i}{\sqrt{2}}a$ in the upper helf complex plane and at the points $\zeta \,=\, \frac{1-i}{\sqrt{2}}a$ and $\zeta \,=\, \frac{-1-i}{\sqrt{2}}a$ in the lower half complex plane. The derivatives of the denominator ξ^4+a^4 at the points $\zeta \,=\, \frac{1+i}{\sqrt{2}}a$ and $\zeta \,=\, \frac{-1+i}{\sqrt{2}}a$ are $4a^3\frac{-1+i}{\sqrt{2}}$ and $4a^3\frac{1+i}{\sqrt{2}}$, respectively, leading to an upper half

plane contribution

$$\frac{2\pi i}{4a^3}\left(\frac{e^{i\left(\frac{1+i}{\sqrt{2}}\right)ax}}{\frac{-1+i}{\sqrt{2}}} + \frac{e^{i\left(\frac{-1+i}{\sqrt{2}}\right)ax}}{\frac{1+i}{\sqrt{2}}}\right)$$

$$= \frac{2\pi i}{4a^3}\left(\frac{-1-i}{\sqrt{2}}\, e^{i\left(\frac{1+i}{\sqrt{2}}\right)ax} + \frac{1+i}{\sqrt{2}}\, e^{i\left(\frac{-1+i}{\sqrt{2}}\right)ax}\right)$$

$$= \frac{2\pi\, e^{-\frac{ax}{\sqrt{2}}}}{2a^3}\left(cos\frac{ax}{\sqrt{2}} + \sin\frac{ax}{\sqrt{2}}\right),\ \ x > 0.$$

The very similar computation of the lower half plane contribution gives the value

$$\frac{2\pi\, e^{\frac{ax}{\sqrt{2}}}}{2a^3}\left(cos\frac{ax}{\sqrt{2}} - \sin\frac{ax}{\sqrt{2}}\right),\ \ x < 0.$$

Going back to the formula for $u(x)$ we have

$$u(x) = -\frac{mg}{2EI\pi}\left(\frac{2\pi\, e^{-\frac{ax}{\sqrt{2}}}}{2a^3}\left(cos\frac{ax}{\sqrt{2}} + \sin\frac{ax}{\sqrt{2}}\right)\right),\ \ x > 0.$$

Simplifying this expression and performing the same operations on the expression valid for $x < 0$, we finally obtain

$$u(x) = \begin{cases} -\frac{mg\, e^{-\frac{ax}{\sqrt{2}}}}{2EIa^3}\left(cos\frac{ax}{\sqrt{2}} + \sin\frac{ax}{\sqrt{2}}\right),\ \ x > 0, \\ -\frac{mg\, e^{\frac{ax}{\sqrt{2}}}}{2EIa^3}\left(cos\frac{ax}{\sqrt{2}} - \sin\frac{ax}{\sqrt{2}}\right),\ \ x < 0. \end{cases}$$

We can complete the definition of $u(x)$ for all $x \in (-\infty, \infty)$ in a continuously differentiable manner by setting

$$u(0) = -\frac{mg}{2EIa^3},\quad u^{'}(0) = 0.$$

Thus the (downward) vertical displacement of the loaded catwalk exhibits oscillating exponential decay toward the nominal constant equilibrium level. The oscillatory behavior of the loaded configuration is, perhaps, not entirely intuitive.